2D Steering Model

Tuesday, November 23, 2021 12:55 PM

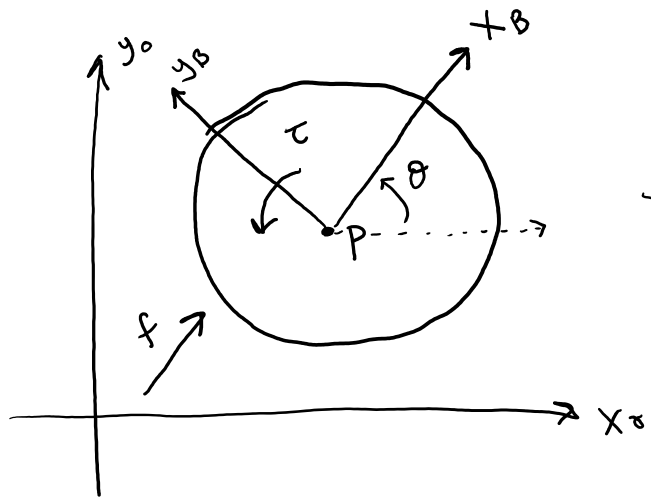
A simple 2D vehicle

→ physics-based control, e.g. move vehicle by applying forces & torques

→ smooth turning, acceleration

Model: "Simple car"

→ 2D disk that can translate in the XY plane & rotate around the Z axis



→ X axis is the fwd dir of the vehicle

→ vehicle always moves "forward", e.g. along its local x-axis

→ apply torque around Z to turn.

→ apply force along the local x-axis to accelerate & decelerate

State: m = mass

I = inertia

p = pos wrt world

θ = heading w.r.t world

Equations of Motion:

Recall for a particle, the eqn of motion were

$$p^{t+\Delta t} = p^t + (\Delta t)\, \dot{p}^t$$

where $p = \begin{bmatrix} x \\ \ldots \end{bmatrix}$ and $\dot{p} = \begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} v \\ a \end{bmatrix} = \begin{bmatrix} v \\ f/m \end{bmatrix}$

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where $f = \sum f_i = ma$

Our vehicle adds state for rotational dynamics

$$p^{t+\Delta t} = p^t + \Delta t \, \dot{p}^t$$

where $p = \begin{bmatrix} x \\ v \\ \theta \\ \dot{\theta} \end{bmatrix}$ and $\dot{p} = \begin{bmatrix} \dot{x} \\ \dot{v} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} v \\ a \\ \omega \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} v \\ f/m \\ \omega \\ \tau/I \end{bmatrix}$

where θ is the rotation around z (radians)

$\dot{\theta}$ is the rotation rate around z (radians/s)

$\rightarrow$ aka angular velocity, ω

x is the position (cm)

$\dot{x} = v$ is the velocity (cm/s)

$\ddot{x} = \dot{v} = a$ is acceleration (cm/s^2)

$\ddot{\theta} = \dot{\omega}$ is angular acceleration (radians/s^2)

and $f = \sum f_i = ma$

$$\tau = \sum \tau_i = I\dot{\omega} + \cancel{\underbrace{\omega \times I\omega}_{\text{co-linear}}}^{0} = I\dot{\omega}$$

In the simple planar case, $I, \theta, \dot{\theta}, \ddot{\theta}$ all scalars

We can simplify further if we model the dynamics in the car's local coordinate system:

$p_B = \begin{bmatrix} 0 \\ \theta \\ \dots \end{bmatrix}$ and $\dot{p}_B = \begin{bmatrix} v \\ \dot{\theta} \\ \dots \end{bmatrix} \leftarrow$ because $x = (0,0,0)^T$ always

$\begin{bmatrix} v \end{bmatrix}$ always in local direction

$$P_B = \begin{bmatrix} 0 \\ \theta \\ v \\ \dot{\theta} \end{bmatrix} \quad \text{and} \quad \dot{P}_B = \begin{bmatrix} v \\ \dot{\theta} \\ f/m \\ \tau/I \end{bmatrix} \leftarrow$$

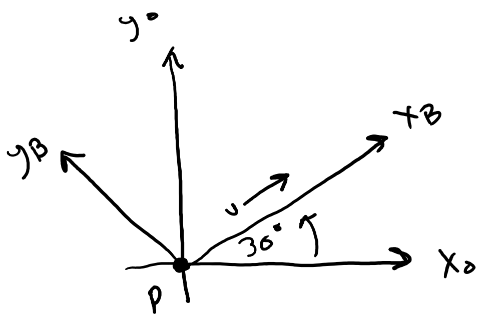
because $x = (0, -)$
f, v always in local direction
(neg ⇒ backwards
pos ⇒ forwards)

All values are scalars now!

After computing $P_B^{t+\Delta t}$, we can compute the global pos & ori of the disc/car.

$$P_{Global}^{t+\Delta t} = \left[\begin{array}{c|c} R_z(\theta^{t+\Delta t}) & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c} P_B^{t+\Delta t} \\ \hline 1 \end{array}\right] + P_{global}^{t}$$

<u>Car Example</u>: Suppose we have a car at initial world pos = (0,0,0) and orientation $\theta = 30°$



local state $P_B = \begin{bmatrix} 0 \\ v \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ 30° \\ 10° \end{bmatrix}$

Suppose we apply a force $f = 5$ & a torque $\tau = 10$ & $\Delta t = 1.0$ s

$m = I = 1.0$
$v = 10$
$\dot{\theta} = 10$

→ car will move fwd along x_B & spin

<u>Step 1</u>: Update local state

$$P_B^{t+\Delta t} = P_B^t + \Delta t\, \dot{P}_B^t = \begin{bmatrix} 0 \\ v^t \\ \theta^t \\ \dot{\theta}^t \end{bmatrix} + \Delta t \begin{bmatrix} v^t \\ f/m \\ \dot{\theta}^t \\ \tau/m \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 10 \\ 30° \\ 10° \end{bmatrix} + (1s) \begin{bmatrix} 10 \\ 5 \\ 10 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ 15 \\ 40° \\ 20° \end{bmatrix} \leftarrow \begin{bmatrix} x \\ v \\ \theta \\ \dot{\theta} \end{bmatrix}$$

Step 2: Update global state

$$p^{t+\Delta t}_{global} = \left[\begin{array}{c|c} R_z(40) & 0 \\ \hline 0 & 1 \end{array}\right] \begin{bmatrix} x \\ \hline 1 \end{bmatrix} + p^t$$

global pos

$$= \left[\begin{array}{c|c} R_z(40) & 0 \\ \hline 0 & 1 \end{array}\right] \begin{bmatrix} 10 \\ 0 \\ 0 \\ \hline 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline 1 \end{bmatrix} = \begin{bmatrix} 7.7 \\ 6.4 \\ 0 \\ \hline 1 \end{bmatrix}$$

global orientation is $R_z(40)$

Note: $\| p^{t+\Delta t} - p^t \| \sim 10$ which matches our intuition because $v = 10$ units/s & $\Delta t = 1$

How can we use forces & torques to control the vehicle?

Idea: Compare current state to desired state + then compute forces that push/pull us towards the desired state

$\boxed{EX}$ Consider translation of a particle

Let $f = \left(\dfrac{p_d - p}{\| p_d - p \|} \right) k_{force}$

We will call $u = \begin{bmatrix} f \\ \tau \end{bmatrix}$ our control vector

The eqns we use to compute u are called control laws

For our car, we will use

- $\rightarrow f$ to track a desired speed
- $\rightarrow \tau$ to track a desired direction

Compute f: $f = m\, k_{v1} (v_d - v)$

note: All values are scalars

<u>Compute f</u> : $f = m\,k_{v1}\,(v_d - v)$

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where v_d is the desired speed

$v_d > v \Rightarrow$ speed up, $a > 0$

$v_d < v \Rightarrow$ slow down, $a < 0$

$v_d = v \Rightarrow f = 0$, stand still

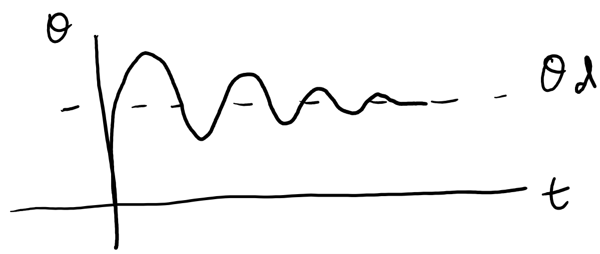
<u>Compute τ</u> : $\tau = I\,(\,k_p\,\underbrace{(\theta_d - \theta)}_{\text{match angle}} - \underbrace{k_{v2}\,\dot{\theta}}_{\text{drag}}\,)$

spring-like dynamics
$\rightarrow \theta_d$ & v_d are like "rest lengths"

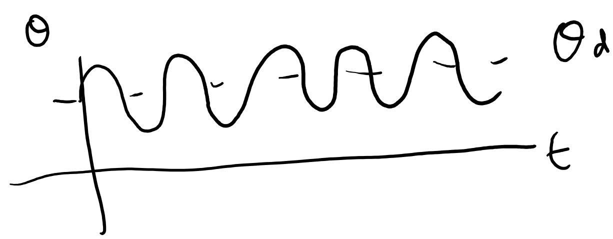
k_{v1} & k_{v2} & k_p are gains. Choice of gains controls the behavior of the system

critically damped

under-damped



pure oscillation (non-convergence)

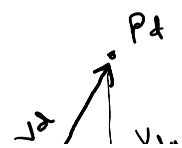


<u>Control law Example</u> :

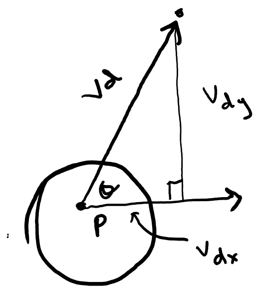
Suppose $p_d = (250, 250, 0)^T$ ← desired position

$p = (0, 0, 0)^T$

$v_B = \theta_B = \dot{\theta}_B = 0$



$$V_B = \theta_B = \dot{\theta}_B = 0$$



$$m = I = 1.0$$

Compute desired direction & speed:

$$V_d = P_d - P = (250, 250, 0)^T$$

$$\|v_d\| \approx 353.55$$

$$\theta_d = \mathrm{atan2}(V_{dy}, V_{dx}) \approx 0.79 \text{ radians}$$

Compute f & τ:

$$f = m \cdot k_{v1}(v_d - v) = k_{v1}(353.55)$$

$$\tau = I\left(k_p(\theta_d - \theta) - k_{v2}\dot{\theta}\right) = k_p(0.79)$$