IK

Sunday, November 7, 2021 2:42 PM

Motivation: IK allows us to adapt a captured motion to new situations

→ ground clamping: clamping the feet to the surface of uneven terrain

→ retarget a motion to a different character

Recall FK:

input local joint rotations → $\boxed{FK}$ → output global positions for end-effectors

IK is like the "opposite" of FK

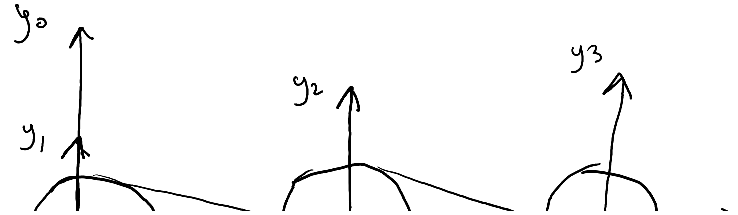
inputs global end-effector joint positions → $\boxed{IK}$ → output local joint rotations

We will study two approaches

1. Analytic Method: breaks problem into 2 geometric subproblems that can be solved directly
2. Cyclic Coordinate Descent (CCD): iterative algorithm that progressively moves the skeleton so the end effector moves towards the desired position

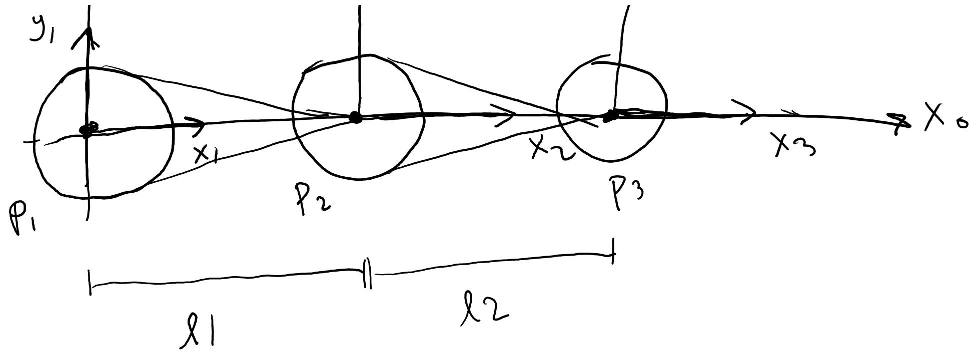
Analytic:

Consider a two-link chain



Goal: Want the global location of $\mathring{p}_3$ to be a

p_3^0 to be a desired location p_d^0,



$$p_d^0 = \underbrace{\left(\begin{array}{c|c} R_1^0 & d_1^0 \\ \hline 0 & 1 \end{array}\right)}_{T_1^0} \underbrace{\left(\begin{array}{c|c} R_2^1 & d_2^1 \\ \hline 0 & 1 \end{array}\right)}_{T_2^1} \underbrace{\left(\begin{array}{c|c} R_3^2 & d_3^2 \\ \hline 0 & 1 \end{array}\right)}_{T_3^2} \underbrace{\left(\begin{array}{c} 0 \\ \hline 1 \end{array}\right)}_{p_3^3}$$

where $d_1^0 = (0,0,0)^T$
$d_2^1 = (l1, 0, 0)^T$
$d_3^2 = (l2, 0, 0)^T$

Our unknowns are R_1^0 & R_2^1

<u>note</u>: R_3^2 doesn't affect the location of p_d so we don't need to compute it

To solve for R_1^0 & R_2^1, we make 2 simplifying assumptions

1. Assume p_2 has 1 DOF (realistic for knees/elbows)
2. Assume p_1 has 2 DOF (twist doesn't change p_d)

<u>Approach</u>: Split problem into 2 subproblems

<u>Step 1</u>: <u>Length</u>

We know the distance between the goal p_d & p_1.
We know the distance between the goal p_d & p_1. We know the length matches.

→ adjust the rotation of p_2 so the length matches.

<u>Step 2</u>: <u>Orientation</u>

Once the length has the desired value, we adjust the rotation of p_1 so it points towards p_d

<u>NOTE</u>: All computations are done in global coordinates

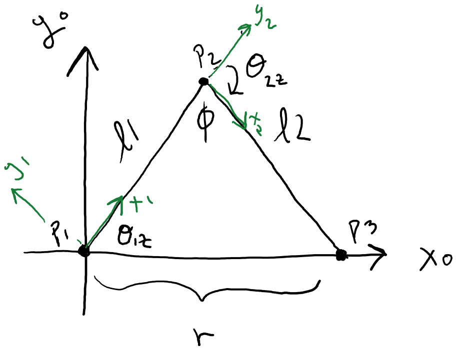
<u>Step 1</u>: Length (details)

We want $\|p_3^0 - p_1^0\| = \|p_d^0 - p_1^0\|$ ↖ joint 1

We want $\|p_3^0 - p_1^0\| = \|p_d^0 - p_1^0\|$

end-effector, joint 1, target, joint 1

Joint 2 has 1 DOF $\Rightarrow$ look at the plane containing $\ell 1$ & $\ell 2$. Then rotate p_2



Law of cosines states that

$$r^2 = (\ell 1)^2 + (\ell 2)^2 - 2(\ell 1)(\ell 2)\cos\phi$$

Let $r = \|p_d - p_1^0\|$

Let ϕ = rotation we want at p_2 (unknown)

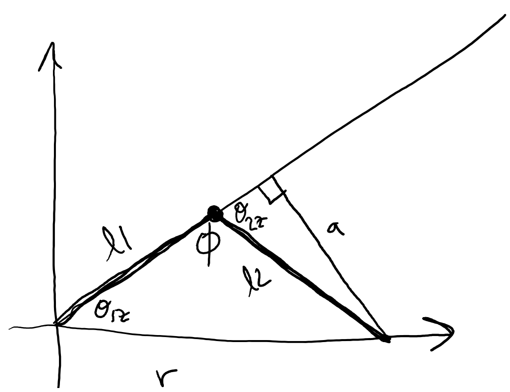
$$\cos\phi = \frac{r^2 - (\ell 1)^2 - (\ell 2)^2}{-2(\ell 1)(\ell 2)}$$

Gotcha: The rotation for p_2 is w.r.t p_1. Therefore, after we compute ϕ, we set the angle to rotate joint 2 as

$$\theta_{2z} = \phi - 180 \quad \text{// negative because we rotate clockwise}$$

Therefore, a $R_z(\theta_{2z})$ rotation will rotate p_2 so that the distance $\|p_3 - p_1\| = \|p_1 - p_d\|$

Now that we have θ_{2z}, lets compute θ_{1z} which will align the direction $p_3 - p_1$ with the x-axis.

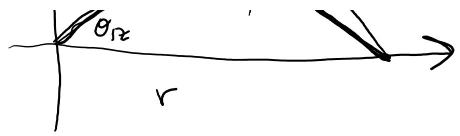


We know $\sin\theta_{1z} = \frac{a}{r}$ and

$$\sin\theta_{2z} = \frac{a}{\ell 2}$$

$$\Rightarrow a = \ell 2 \sin\theta_{2z}$$

$$\Rightarrow \sin(\theta_{1z}) = -\,\underline{\ell 2 \sin\theta_{2z}}$$



$$\Rightarrow \sin(\theta_{1z}) = -\frac{l2 \sin\theta_{2z}}{r}$$

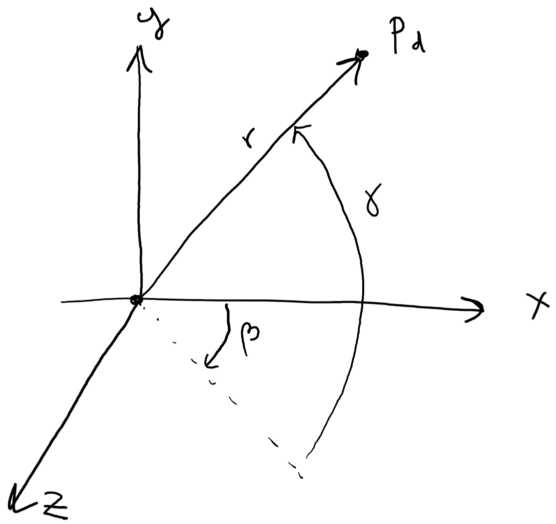
minus because we rotate clockwise

Step2: Aiming the arm towards the target

Idea: Make the direction $(p_3^0 - p_1^0)$ point towards the target

(We will cover 2 methods, one based on euler angles & one based on tangent)

Euler angle approach computes a rotation around Y (up direction, heading), and a rotation around Z (pitch).



Let β = angle around Y

γ = angle around Z ← this is in addition to θ_{1z}

α = angle around X ← ϕ because we don't need/want twist

Using transformations:

$$P_d^0 = \left(\begin{array}{c|c} R_y(\beta) & 0 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} R_z(\gamma) & 0 \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} R_x(\alpha) & 0 \\ \hline 0 & 1 \end{array}\right)\begin{pmatrix} r \\ 0 \\ 0 \\ \hline 1 \end{pmatrix}$$

note: $$\begin{bmatrix} r \\ 0 \\ 0 \\ \hline 1 \end{bmatrix} = \left[\begin{array}{c|c} R_z(\theta_{1z}) & \begin{pmatrix} l1 \\ 0 \\ 0 \end{pmatrix} \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} R_z(\theta_{2z}) & \begin{pmatrix} l2 \\ 0 \\ 0 \end{pmatrix} \\ \hline 0 & 1 \end{array}\right]\begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline 1 \end{bmatrix}$$

note: $R_x(\alpha) = I$

$\therefore$ $\begin{bmatrix} r\cos\beta\cos\gamma \end{bmatrix}$ ← we know P_d^0 and r

$$P_d^0 = \begin{bmatrix} r\cos\beta\cos\gamma \\ r\sin\gamma \\ -r\sin\beta\cos\gamma \end{bmatrix}$$ ← we kn- P_d^0 and r

$$r\sin\gamma = (P_d^0)_y \qquad -\tan\beta = \frac{-r\sin\beta\cos\gamma}{r\cos\beta\cos\gamma} = \frac{(P_d^0)_z}{(P_d^0)_x}$$

$$\gamma = \operatorname{asin}\left(\frac{(P_d^0)_y}{r}\right) \qquad \Rightarrow \beta = \operatorname{atan2}\left(-(P_d^0)_z, (P_d^0)_x\right)$$

Summary: Putting everything together

Let R_2^1 = rotation at p_2 = $R_z(\theta_{2z})$

Let R_1^0 = rotation at p_1 = $R_y(\beta)R_z(\gamma)R_z(\theta_{1z})$

$\boxed{EX}$ Suppose we have a 2 link chain where each joint extends forward along X.

$l1 = 3$

$l2 = 2$

$P_d = (-3, \sqrt{7}, 0)^T$

Solve for R_1^0 and R_2^1.

(a) What is the length r?

$$r = \|p_d - P_1\| = \left\| \begin{pmatrix} -3 \\ \sqrt{7} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\| = \sqrt{(-3)^2 + (\sqrt{7})^2} = \sqrt{9+7} = 4$$

(b) What is ϕ?

$$\cos\phi = \frac{r^2 - (l1)^2 - (l2)^2}{-2(l1)(l2)} \quad \text{where } l1 = 3,\ l2 = 2,\ r = 4$$

$$\cos\phi = \frac{(4)^2 - (3)^2 - (2)^2}{-2(3)(4)} = \frac{3}{-12} = -\frac{1}{4}$$

$$\Rightarrow \phi \sim 1.8235$$
$$\sim 104°$$

(c) What is θ_{2z}?

$\sim 104°$

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$$\theta_{2z} = \phi - 180 \sim -76 \sim -1.3181$$

(d) What is θ_{1z}?

$$\sin\theta_{1z} = \frac{-(\ell 2)\sin\theta_{2z}}{r}$$

$$= \frac{-(2)\sin(-75)}{4} \Rightarrow \theta_{1z} \sim 29°$$

(e) What is β, γ?

$$\gamma = \operatorname{asin}\left(\frac{(P_d^0)_y}{r}\right) = \operatorname{asin}\left(\frac{\sqrt{7}}{4}\right) \sim 41.4$$

$$\beta = \operatorname{atan2}(0, -3) = 180$$

To test, plug the local rotations into the kinematic eqns for the 2-link chain

$$P_3^0 = \left[\begin{array}{c|c} R_1^0 & 0 \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} R_2^1 & d_2^1 \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} R_3^2 & d_3^2 \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right]$$

where $R_1^0 = R_y(180)\, R_z(41.4)\, R_z(29)$

$R_2^1 = R_z(-76)$

$d_2^1 = (\ell 1, 0, 0)^T = (3, 0, 0)^T$

$d_3^2 = (\ell 2, 0, 0)^T = (2, 0, 0)^T$

P_3^0 will be located at P_d^0 (try it!)