Transforms (cont.)

4x4 matrix transforms are called
homogeneous transformations because the operate on homogeneous coordinates

<u>Notes</u>: homogeneous transforms can be represented as 2x2 block matrices

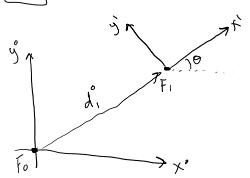
$$\left[\begin{array}{c|c} R & d \\ \hline 0 & 1 \end{array}\right]$$

homogeneous coordinates can be represented as 2x1 block matrices

points $\left[\begin{array}{c} p \\ \hline 1 \end{array}\right]$ and vector $\left[\begin{array}{c} v \\ \hline 0 \end{array}\right]$

We can combine these block representations the same as regular matrix & vector arithmetic

$\boxed{\text{Ex}}$ Consider our frame of reference from last class



The transformation T_1^0 which converts coordinates from frame 1 (F_1) to frame ϕ (F_0) is

$$T_1^0 = \left[\begin{array}{c|c} R_1^0 & d_1^0 \\ \hline 0 & 1 \end{array}\right]$$

Suppose $R_1^0 = R_Z(45)$ and $d_1^0 = (5,2,0)^T$.
What is the location of the origin of F_1 w.r.t. F_0?

<u>Aside</u>: F_1 is an example of a <u>local coordinate</u> system

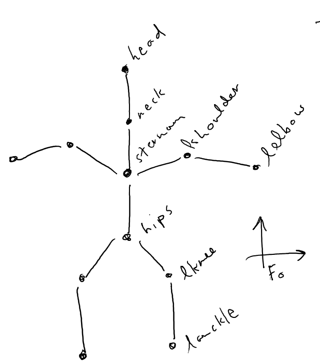
Let $p^1 = (0,0,0)^T$, then $p^0 = T_1^0 p^1$

$$= \left[\begin{array}{c|c} R_1^0 & d_1^0 \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c} p^1 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} R_1^0 p^1 + d_1^0 \\ \hline 0 \cdot p^1 + 1 \end{array}\right] = \left[\begin{array}{c} R_1^0 p^1 + d_1^0 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} R_Z(45)\begin{pmatrix}0\\0\\0\end{pmatrix} + \begin{pmatrix}5\\2\\0\end{pmatrix} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} \begin{pmatrix}5\\2\\0\end{pmatrix} \\ \hline 1 \end{array}\right]$$

What is the position of $p^1 = (1,0,0)^T$ w.r.t. F_0?

$$p^0 = T_1^0 p^1 = \left[\begin{array}{c} R_1^0 p^1 + d_1^0 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} R_Z(45)\begin{pmatrix}1\\0\\0\end{pmatrix} + \begin{pmatrix}5\\2\\0\end{pmatrix} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} \begin{pmatrix}\sqrt{2}/2\\\sqrt{2}/2\\0\end{pmatrix} + \begin{pmatrix}5\\2\\0\end{pmatrix} \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} \frac{\sqrt{2}}{2} + 5 \\ \sqrt{2}/2 + 2 \\ 0 \\ \hline 1 \end{array}\right]$$

<u>Articulated Characters</u>



→ Represented as a hierarchy of joints (aka bones)
- → hierarchy is called a <u>skeleton</u>
- → root of the hierarchy is usu. the hips
  - → the root transform is usu. w.r.t. to the world transform
- → each joint stores a transform w.r.t to its parent
- → joints with no children are called <u>end effectors</u>

object can move

→ joints with no children are called end eff

Degrees of freedom (DOF): the "# ways" an object can move

$\boxed{EX}$ the "hips" have 6 DOF: translation in XYZ
rotation around X Y Z axes

$\boxed{EX}$ the "lknee" might have 3 DOF: rotation around XYZ

Joints: Several types of joints

1. Ball joint (ex. shoulder) ← all joints in our class are these (3 DOF, rotation around XYZ)
2. Hinge joint (ex. elbow) ← 1 DOF joint (rotational)
3. Prismatic joint (ex. selfie stick) ← 1 DOF (translational)

Implementation details (base code):

Skeleton maintains tree of joints

Each joint stores

2 transforms: ① local2parent (F_i^j): converts from the joint's frame to its parent frame

② local2global (F_i^0) converts from the joint's frame to the global frame

name (ex. "hips")
id (0, 1, 2, ..., numJoints-1)

ptr to parent
list of children ptrs.

Kinematics: refers to the motion of objects over time

Forward Kinematics: Process of computing global positions of each joint given the current state of the skeleton.

The current state of a skeleton is called a "pose".

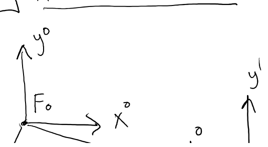
Idea: A pose contains a value for every DOF in the character.

$\boxed{EX}$ The above skeleton has $6 + 11 * 3$ DOFs (root + other joints)

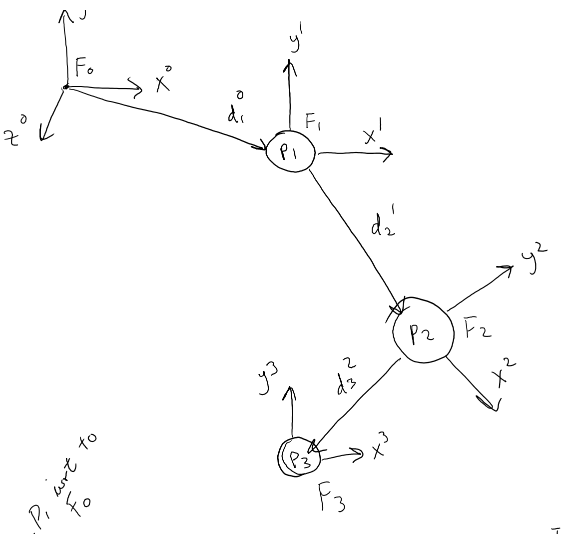
Therefore, our pose have 39 values in it

$$\Theta = [\underbrace{root_x, root_y, root_z}_{\text{root translation}}, \underbrace{root\,\theta_x, root\,\theta_y, root\,\theta_z}_{\text{root rot}}, \underbrace{lknee\,\theta_x, lknee\,\theta_y, \ldots}_{\text{all values in local coordinates}}]$$

$\boxed{EX}$ Kinematic Chain



Goal: Compute the global positions of P_1, P_2, P_3 given the local transforms at P_1 P_2 P_3, e.g.



p_1, p_2, p_3 given the local transforms at p_1 p_2 p_3, e.g.

T_1^0, T_2^1, T_3^2

Approach: Compute T_1^0, T_2^0, and T_3^0 to get the global positions of p_1 p_2 & p_3.

p_1 wrt to F_0

$$p_1^0 = T_1^0 p_1^1 , \text{ where } p_1^1 = (0,0,0,1)^T$$

$$p_2^0 = T_2^0 p_2^2 = T_1^0 T_2^1 p_2^2 \leftarrow p_2^2 = (0,0,0,1)^T$$

$$p_3^0 = T_3^0 p_3^3 = T_1^0 T_2^1 T_3^2 p_3^3$$

$\boxed{\text{EX}}$ Suppose $d_1^0 = (1,-1,1)^T$, $d_2^1 = (2,0,0)^T$, $d_3^2 = (2,-1,0)^T$

$R_1^0 = R_z(-45)$

$R_2^1 = R_z(-30)$

$R_3^2 = I$

ϕ is a 3x1 vector

$$p_3^0 = T_3^0 p_3^3 = T_1^0 T_2^1 T_3^2 p_3^3$$

$$= \left[\begin{array}{c|c} R_1^0 & d_1^0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} R_2^1 & d_2^1 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} R_3^2 & d_3^2 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c} \phi \\ \hline 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} R_3^2 & d_3^2 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c} \phi \\ \hline 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R_1^0 R_2^1 R_3^2 & R_1^0 R_2^1 d_3^2 + R_1^0 d_2^1 + d_1^0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c} \phi \\ \hline 1 \end{array}\right]$$

= do later

(FK)
Forward Kinematics Alg: generalizes process we just did w/ the kinematic chain to any skeleton

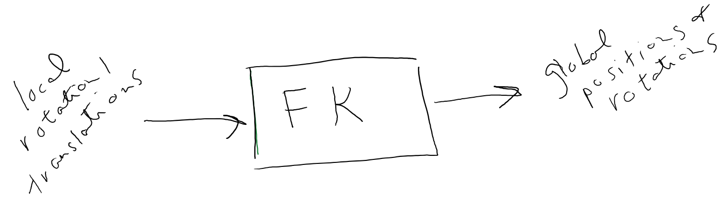
FK: Recursively compute local2global F_j^0 for each joint j

Idea: Do a tree traversal on each joint. Start at root

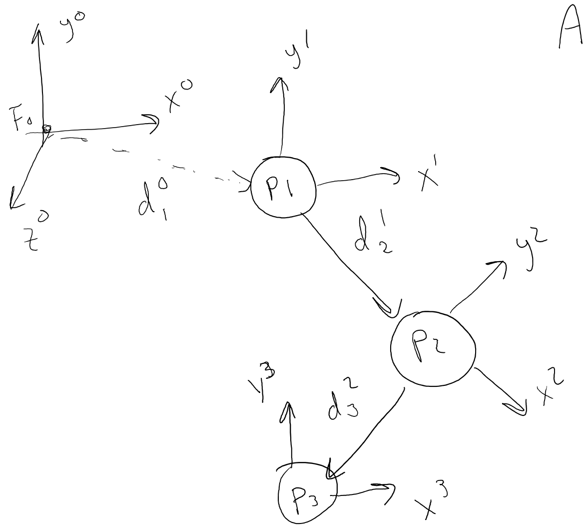
```
Joint :: fk()
  if (mParent != NULL)
    local2global = (mParent -> local2global) * local2parent
  else
    local2global = local2parent;  //root

  for each child
    child -> fk()
```

$$\text{local2global} = \underbrace{(\text{mParent} \to \text{local2global})}_{F_i^0} * \underbrace{\text{local2parent}}_{F_j^i}$$



Kinematic Chain Revisited:



Assume we are given local displacements & rotations at each joint, e.g.

$d_1^0,\ d_2^1,\ d_3^2$ ← displacement / offset / translation

$R_1^0,\ R_2^1,\ R_3^2$ ← orientations (usu euler angles, or quats, but 3x3 matrices here)

Q: What is the local2global transform, T_1^0, for joint 1?

$$T_1^0 = \left[\begin{array}{c|c} R_1^0 & d_1^0 \\ \hline 0 & 1 \end{array}\right]$$

Q: What is the <u>local2global</u> transform of joint 3?

$$T_3^0 = T_1^0\ T_2^1\ \underline{T_3^2} = \left[\begin{array}{c|c} R_1^0 & d_1^0 \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} R_2^1 & d_2^1 \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} R_3^2 & d_3^2 \\ \hline 0 & 1 \end{array}\right]$$

... frame?

$T_3 = T_1 \quad T_2 \quad T_3 \quad \left[0 \; 1 \right] \left[0 \; 1 \; \cdots \right]$

Q: What is the coordinate of p_3^3 in joint 2's frame?

position of joint 3 w.r.t frame 3

$$p_3^2 = T_3^2 \; p_3^3$$

$$= \left[\begin{array}{c|c} R_3^2 & d_3^2 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} R_3^2 \cdot \phi + d_3^2 \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} d_3^2 \\ \hline 1 \end{array}\right]$$

Q: What is the coordinate p_2^2 in joint 3's frame?

$$p_2^3 = \underbrace{T_2^3}_{\text{don't have this but have } T_3^2} p_2^2 = \left(T_3^2\right)^{-1} p_2^2$$

$$\text{where } \left(T_3^2\right)^{-1} = \left(\left[\begin{array}{c|c} R_3^2 & d_3^2 \\ \hline 0 & 1 \end{array}\right] \right)^{-1}$$

$$= \left(\begin{array}{c|c} \left(R_3^2\right)^T & -\left(R_3^2\right)^T d_3^2 \\ \hline 0 & 1 \end{array}\right)$$

$$\left[\begin{array}{c|c} R_3^2 & d_3^2 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} A & b \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{c|c} R_3^2 A & R_3^2 b + d_3^2 \\ \hline 0 & 1 \end{array}\right]$$

$\rightarrow$ Let $A = \left(R_3^2\right)^T$

$$R_3^2 b + d_3^2 = 0$$

$$\rightarrow R_3^2 b = -d_3^2$$

$$\rightarrow b = \left(R_3^2\right)^T \left(-d_3^2\right)$$

Q: What is the world origin w.r.t. joint 3's frame?

$$p_0^3 = \underbrace{T_0^3 \, p_0^0} = \left(T_0^3\right)^{-1} p_0^0$$

$$= \left[\begin{array}{c|c} \left(R_1^0 R_2^1 R_3^2\right)^{-1} & \left(R_1^0 R_2^1 R_3^2\right)^{-1} \left(-R_1^0 R_2^1 d_3^2 - R_1^0 d_2^1 - d_1^0\right) \\ \hline 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{c|c} & \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R_2^3 R_1^2 R_0^1 & -R_2^3 d_3^2 - R_2^3 R_1^2 d_2^1 - R_2^3 R_1^2 R_0^1 d_1^0 \\ \hline 0 & 1 \end{array}\right]$$

Skeletons & Joints Revisited:

Recall: A pose Θ is a vector of values corresponding to the DOF of the skeleton.

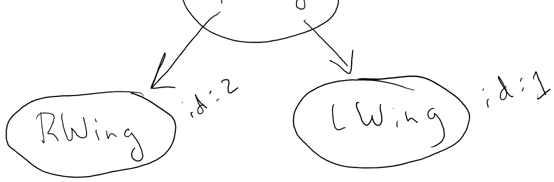
$\boxed{\text{Ex}}$ A biped w 6 DOF at the root & 3 at all other joints

$$\Theta = (\underbrace{x_0^0\ y_0^0\ z_0^0}_{\substack{\text{translation}\\ \text{joint } \emptyset,\\ \text{wrt to}\\ \text{world}}}\ \underbrace{\theta_{0x}^0\ \theta_{0y}^0\ \theta_{0z}^0}_{\substack{\text{rotation}\\ \text{joint } \emptyset\\ \text{w.r.t}\\ \text{world}}},\ \underbrace{\theta_{1x}\ \theta_{1y}\ \theta_{1z}}_{\text{rotation joint 1}}, \ldots,\ \underbrace{\theta_{Nx}\ \theta_{Ny}\ \theta_{Nz}}_{\text{joint } N})$$

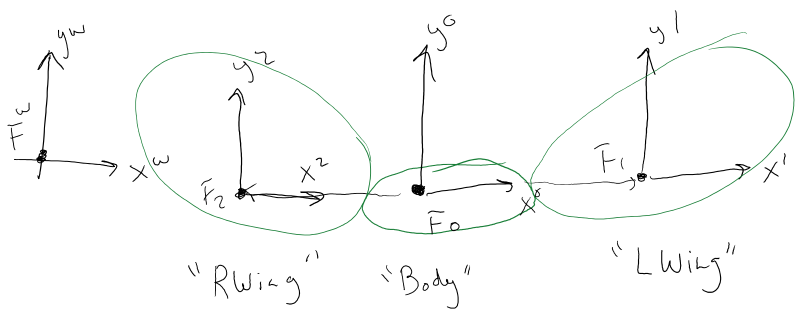
To animate an articulated character (e.g. one w/ poseable limbs), we copy the pose into the transform of each joint & call FK

$\boxed{\text{Example}}$ Butterfly

Skeleton Hierarchy:



Define how each body part relates to the other



Goal: Associate geometry w/ each transform.

Common Approach: Scene Graph

an entire scene,

Common Approach: Scene Graph

→ a tree data structure that represents an entire scene,

→ nodes represent scene objects (light, primitives, etc.)

→ each node has a transform

→ edges represent parent-child frame relationships