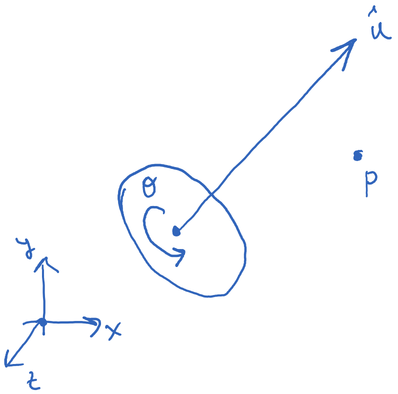
Angle/Axis rotation representations

Tuesday, October 5, 2021 12:54 PM

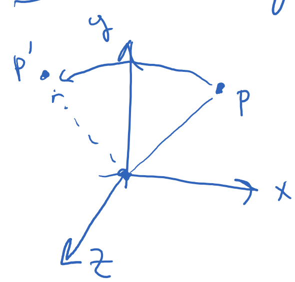
For euler angles, we rotate around X, Y, & Z axes in some order.

→ but what about an arbitrary axis?

let $\hat{u}$ be a unit vector to rotate around



Recall: rotating around z



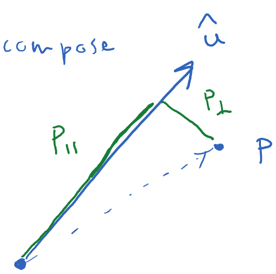
* p is in XY plane, ⊥ to the z axis

* p is not w.r.t to a plane ⊥ to $\hat{u}$

Approach:

1. Decompose p into 2 parts
   - (a) One $p_\perp$ which is perpendicular to u
   - (b) One, $p_\parallel$, which parallel to u
2. Rotate $p_\perp$
3. Put $p_\perp$ + $p_\parallel$ "back together"

Step 1: Decompose



$P_\parallel$ = projection of p onto $\hat{u}$

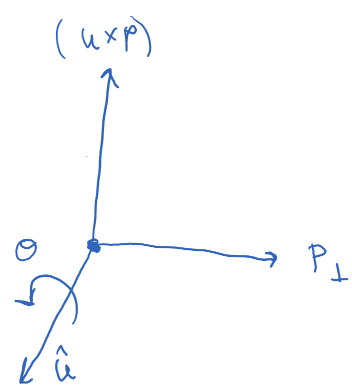
$$= \left(p \cdot \frac{\hat{u}}{\|\hat{u}\|}\right)\frac{\hat{u}}{\|\hat{u}\|} \qquad \|\hat{u}\| = 1$$

$$= (p \cdot u)\, p$$

$$P_\perp = p - P_\parallel = p - (p \cdot u)u$$

Step 2: Rotate

$$P_\perp' = \cos\theta\, P_\perp + \sin\theta\,(u \times p)$$



Step 3: Put $P_\perp'$ & $P_\parallel$ together

$$p' = p_{\parallel} + p'_{\perp} = (p \cdot u)u + \cos\theta \, p_{\perp} + \sin\theta \, (u \times p)$$

$$= \cos\theta \, p + (1 - \cos\theta)(p \cdot u)u + \sin\theta \, (u \times p)$$

<u>NOTE</u>: We usually use the above eqn in matrix form

$$R_u(\theta) = \begin{bmatrix} c\theta + (1 - c\theta)u_x^2 & -s\theta u_z + (1 - c\theta) u_x u_y & u_y s\theta + u_x u_z (1 - c\theta) \\ -u_z s\theta + (1 - c\theta) u_x u_y & c\theta + (1 - c\theta) u_y^2 & -u_x s\theta + (1 - c\theta) u_y u_z \\ s\theta u_y + (1 - c\theta) u_x u_y & -u_x s\theta + (1 - c\theta) u_y u_z & c\theta + (1 - c\theta) u_z^2 \end{bmatrix}$$