Euler angles

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Euler Angles

- Easier to work w/ than matrices
- Series of 3 rotations around each axis
- Can be any order, e.g. XYZ, ZXY, etc.
- Sometimes talked about in terms of
  - **yaw / heading**: rotation around UP (turn)
  - **pitch**: rotation around LEFT or RIGHT (bow) ← for this class
  - **roll**: rotation around FORWARD (twist)

$\boxed{\text{EX}}$ The ZYX euler rotation corresponds to this matrix

$$R_{ZYX}(\alpha, \theta, \beta) = R_Z(\beta)\, R_Y(\theta)\, R_X(\alpha)$$

rotation around X, Y, Z

vec3 euler(α, θ, β); // our codebase

Multiply the matrices on the LHS to get the corresponding rotation matrix

$$R_{ZYX}(\alpha,\theta,\beta) = \underbrace{\begin{bmatrix} c\beta & -s\beta & 0 \\ s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_Z} \underbrace{\begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}}_{R_Y} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha & -s\alpha \\ 0 & s\alpha & c\alpha \end{bmatrix}}_{R_X}$$

$$= \begin{bmatrix} c\theta c\beta & s\alpha c\theta c\beta - c\alpha s\beta & s\alpha s\beta + c\alpha s\theta c\beta \\ c\theta s\beta & s\alpha s\theta s\beta + c\alpha s\beta & s\beta s\theta c\alpha - c\beta s\alpha \\ -s\theta & c\theta s\alpha & c\theta c\alpha \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\begin{bmatrix} r_{21} & & \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Take a look at R_{ZYX}. Given any rotation matrix R, how can we get the euler angles back?

Step 1: Start w/ simplest term

$$-s\theta = r_{31} \Rightarrow a\sin(-r_{31}) = \theta \quad // \text{ Y rotation}$$

Step 2: To get β: $\frac{r_{21}}{r_{11}} = \frac{c\theta s\beta}{c\theta c\beta} = \frac{s\beta}{c\beta} = \tan\beta \Rightarrow \operatorname{atan2}(r_{21}, r_{11}) = \beta$

To get α: $\frac{r_{32}}{r_{33}} = \frac{\cancel{c\theta} s\alpha}{\cancel{c\theta} c\alpha} = \tan\alpha \Rightarrow \operatorname{atan2}(r_{32}, r_{33}) = \alpha$

But what about when $\theta = \pi/2$ or $-\pi/2$?

→ $s\theta$ will be either 1 or -1

→ our helpfull terms will disappear!

We get

$$\theta = \frac{\pi}{2} \rightarrow \begin{pmatrix} 0 & s\alpha c\beta - c\alpha s\beta & s\alpha s\beta + c\alpha c\beta \\ 0 & s\alpha s\beta + c\alpha c\beta & s\beta c\alpha - c\beta s\alpha \\ -1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & s(\alpha-\beta) & c(\alpha-\beta) \\ 0 & c(\alpha-\beta) & -s(\alpha-\beta) \\ -1 & 0 & 0 \end{pmatrix}$$

when "middle angle" is ± 90, we "lose a degree of freedom"

cos/sin addition identities

$$\theta = \frac{-\pi}{2} \rightarrow \begin{pmatrix} 0 & -s\alpha c\beta - c\alpha s\beta & s\alpha s\beta - c\alpha c\beta \\ 0 & -s\alpha s\beta + c\alpha c\beta & -s\beta c\alpha - c\beta s\alpha \\ 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -s(\alpha+\beta) & -c(\alpha+\beta) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \underline{-s(\alpha+\beta)} & \underline{-c(\alpha+\beta)} \\ 0 & c(\alpha+\beta) & -s(\alpha+\beta) \\ 1 & 0 & 0 \end{pmatrix}$$

How to solve for α & β?

Let $\beta = 0$ (it can be anything)

then $\alpha = \text{atan2}(r_{12}, r_{13})$ because $\dfrac{r_{12}}{r_{13}} = \dfrac{-s(\alpha+\cancel{\beta}^{0})}{-c(\alpha+\cancel{\beta}^{0})} = \tan(\alpha+\cancel{\beta}^{0})$

<u>NOTE</u>: $\text{atan2}(r_{12}, r_{13}) = \alpha + \beta$. We then have to decide how to split the angle between $\alpha + \beta$.

ex. if $\alpha + \beta = 135$, then $\alpha = 5$, $\beta = 130$

Limitations of Euler Angles

<u>Gimbal Lock</u>: Lose a degree of freedom (e.g. an axis of rotation) when the middle angle is $\pi/2$.

$\rightarrow$ one axis rotates onto another

<u>Not unique</u>: Different euler angles may refer to the same rotation

$\boxed{\text{Example}}$ Suppose a rotation was constructed w/ a seqn of 2 rotations XZ

① Find $R_{XZ} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cx & -sx \\ 0 & sx & cx \end{bmatrix} \begin{bmatrix} cz & -sz & 0 \\ sz & cz & 0 \\ 0 & 0 & 1 \end{bmatrix}$

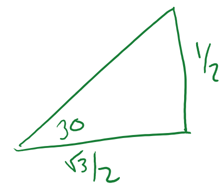
$$= \begin{bmatrix} cz & -sz & 0 \\ cx\,sz & cx\,cz & -sx \\ sx\,sz & sx\,cz & cx \end{bmatrix}$$

② S... $R = \begin{bmatrix} \overset{cz}{0.9659} & \overset{-sz}{-0.2588} & 0 \\ \dots & \dots & \overset{-sx}{-0.5} \end{bmatrix}$

② Suppose

$$R_{xz} = \begin{bmatrix} 0.9659 & -0.2588 & 0 \\ 0.2241 & 0.8365 & -0.5 \\ 0.1294 & 0.4830 & 0.8660 \end{bmatrix} \begin{array}{l} cz \\ -sx \\ cx \end{array}$$

Extract the x & z angles.



$$-0.5 = -sx \Rightarrow \frac{1}{2} = sx \Rightarrow \operatorname{asin}\left(\frac{1}{2}\right) = 30$$

$$\text{in radians } 30 \cdot \frac{\pi}{180} = \pi/6$$

$$0.9659 = cz \Rightarrow \operatorname{acos}(0.9659) = z \Rightarrow z = \sim 15$$