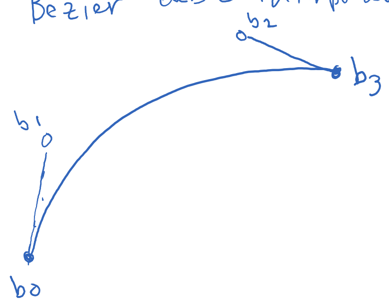
# Hermite

Thursday, September 23, 2021 12:58 PM

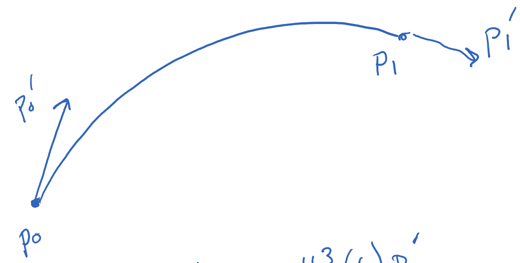
Bezier cubic interpolation




$$p(t) = (1-t)^3 b_0 + 3t(1-t)^2 b_1 + 3t^2(1-t) b_2 + t^3 b_3$$

Uses 4 points

Hermite cubic interpolation




$$p(t) = H_0^3(t) p_0 + H_1^3(t) p_0' + H_2^3(t) p_1' + H_3^3(t) p_1$$

Uses 2 points & 2 directions

Hermite splines are based on Bezier curves

Recall: $p(t) = b_0 B_0^3(t) + b_1 B_1^3(t) + b_2 B_2^3(t) + b_3 B_3^3(t)$

We want to rearrange terms to get

$$p(t) = p_0 H_0^3(t) + \underline{p_0'} H_1^3(t) + \underline{p_1'} H_2^3(t) + p_1 H_3^3(t)$$

slopes at p0 & p1

Recall the slopes at the start & end points of each segment

$$p'(0) = \text{start slope} = p_0' = 3(b_1 - b_0) \Rightarrow b_1 = \tfrac{1}{3} p_0' + b_0$$

$$p'(1) = \text{end slope} = p_1' = 3(b_3 - b_2) \Rightarrow b_2 = -\tfrac{1}{3} p_1' + b_3$$

Substitute our expressions for $b_1$ & $b_2$

$$p(t) = b_0 B_0^3(t) + \left[\tfrac{1}{3} p_0' + b_0\right] B_1^3(t) + \left(-\tfrac{1}{3} p_1' + b_3\right) B_2^3(t) + b_3 B_3^3(t)$$

$$= b_0 \underbrace{\left(B_0^3(t) + B_1^3(t)\right)}_{H_0^3(t)} + p_0' \underbrace{\left(\tfrac{1}{3} B_1^3(t)\right)}_{H_1^3(t)} + p_1' \underbrace{\left(-\tfrac{1}{3} B_2^3(t)\right)}_{H_2^3(t)} + \underbrace{\left[B_2^3(t) + B_3^3(t)\right]}_{H_3^3(t)} b_3$$

Recall that $B_0^3(t) = (1-t)^3$ $\quad B_1^3(t) = 3t(1-t)^2$

$B_3^3(t) = t^3$ $\quad B_2^3(t) = 3t^2(1-t)$

$H_0^3(t) = B_0^3(t) + B_1^3(t)$ $\qquad H_2^3(t) = -t^2 + t^3$

$$H_0^3(t) = B_0^3(t) + B_1^3(t) = 1 - 3t^2 + 2t^3$$

$$H_2^3(t) = -t^2 + t^3$$

$$H_3^3(t) = 3t^2 - 2t^3$$

$$H_1^3(t) = t(1-t)^2$$

Hermite splines allow us to solve for cubic splines that have $C^2$ continuity (recall: catmull-rom splines had $C^1$ continuity)

→ usefull for ensuring both the slope (shape) & speed at each data point is continuous

<u>Goal</u>: Given points from the user, we wish to solve for slopes that guarantee $C^2$ continuity

how? Compute an expression for the second derivatives, $P''_{i+1}(0)$ & $P''_i(1)$, at each point

↑ start of "next" segment

↑ end of "prev" segment

<u>Step 1</u>: Compute $P''_i(t)$

(1a) Compute the first derivative

$$P'_i(t) = P_i H'^3_0(t) + P'_i H'^3_1(t) + P'_{i+1} H'^3_2(t) + P_{i+1} H^3_3(t)$$

$$H'^3_0 = \frac{d}{dt}\left(1 - 3t^2 + 2t^3\right) = -6t + 6t^2$$

$$H'^3_1 = \frac{d}{dt}\left(t - 2t^2 + t^3\right) = 1 - 4t + 3t^2$$

$$H'^3_2 = \frac{d}{dt}\left(-t^2 + t^3\right) = -2t + 3t^2$$

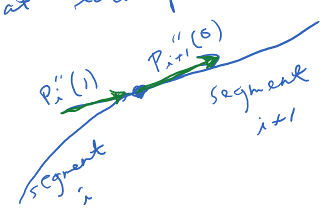
$$H'^3_3 = \frac{d}{dt}\left(3t^2 - 2t^3\right) = 6t - 6t^2$$

$$P'_i(t) = P_i(-6t + 6t^2) + P'_i(1 - 4t + 3t^2) + P'_{i+1}(-2t + 3t^2) + P_{i+1}(6t - 6t^2)$$

(1b) Then, $P''_i(t) = P_i(-6 + 12t) + P'_i(-4 + 6t) + P'_{i+1}(-2 + 6t) + P_{i+1}(6 - 12t)$

<u>Step 2</u>: Set the second derivatives equal at each pt

$$\underbrace{P''_{i+1}}_{\text{segment}}(0) = P''_i(1)$$



$$p''_{i+1}(0) = p_{i+1}(-6) + p'_{i+1}(-4) + p'_{i+2}(-2) + p_{i+2}(6)$$

$$p''_i(1) = p_i(-6+12) + p'_i(-4+6) + p'_{i+1}(-2+6) + p_{i+1}(6-12)$$

$$= p_i(6) + p'_i(2) + p'_{i+1}(4) + p_{i+1}(-6)$$

If $p''_i(1) = p''_{i+1}(0)$, then

$$6p_i + 2p'_i + 4p'_{i+1} - 6p_{i+1} = -6p_{i+1} - 4p'_{i+1} - 2p'_{i+2} + 6p_{i+2}$$

Put unknowns on RHS & knowns on the LHS

$$6p_i - \cancel{6p_{i+1}} + \cancel{6p_{i+1}} - 6p_{i+2} = -2p'_i \underline{- 4p'_{i+1} - 4p'_{i+1}} - 2p'_{i+2}$$

$$6(p_i - p_{i+2}) = -2p'_i - 8p'_{i+1} - 2p'_{i+2} \quad \text{// divide by } -2$$

$$3(p_{i+2} - p_i) = p'_i + 4p'_{i+1} + p'_{i+2}$$

Now, we can formulate a system of equations for each slope.

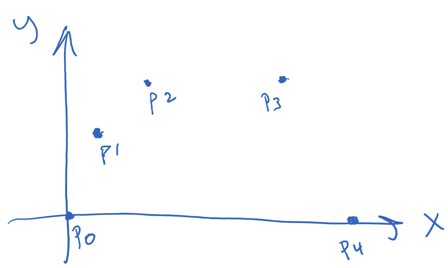
We want something like

$$\underset{\text{knowns}}{P} = \underset{\text{matrix of coefficients}}{A}\ \underset{\text{matrix of unknowns}}{P'} \longrightarrow \begin{bmatrix} \vdots \\ 3(p_{i+2}-p_i) \\ \vdots \end{bmatrix} = \begin{bmatrix} & \vdots & \\ \cdots 1 & 4 & 1 \cdots \\ & \vdots & \end{bmatrix} \begin{bmatrix} \vdots \\ p'_i \\ p'_{i+1} \\ p'_{i+2} \\ \vdots \end{bmatrix}$$

To solve, we compute $A^{-1}P = P'$

$\boxed{\text{EX}}$ Suppose we have 5 points

$p_0 = (0,0)$
$p_1 = (1,2)$
$p_2 = (3,3)$
$p_3 = (6,3)$
$p_4 = (8,0)$




$$3(p_2 - p_0) = p'_0 + 4p'_1 + p'_2 + 0p'_3 + 0p'_4$$

$$3(p_3 - p_1) = p'_1 + 4p'_2 + p'_3$$

$$\underbrace{3(p_4 - p_2)}_{P} = \underbrace{p'_2 + 4p'_3 + p'_4}_{AP'}$$

$$\underset{\substack{3\times 2 \\ \text{matrix}}}{\begin{bmatrix} (9,9) \\ (15,3) \\ (15,-9) \end{bmatrix}} = \underset{\substack{3\times 5 \\ \text{matrix}}}{\begin{bmatrix} 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \end{bmatrix}} \underset{5\times 2}{\begin{bmatrix} p'_0 \\ p'_1 \\ p'_2 \\ p'_3 \\ p'_4 \end{bmatrix}}$$

Problem: We have 5 unknowns but only 3 eqns.

Need endpoint conditions: Two approaches

→ Clamped

→ Natural

Clamped: Use a hard-coded slope (user-specified or constant)

$$\begin{bmatrix} v_0 \\ 3(p_2 - p_0) \\ 3(p_3 - p_1) \\ 3(p_4 - p_2) \\ v_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_0' \\ p_1' \\ p_2' \\ p_3' \\ p_4' \end{bmatrix}$$

$p_0' = v_0$

$p_4' = v_1$

$v_0$ & $v_1$ might be $(1,0)$

Natural: Let $p_0''(0) = p_{n-1}''(1) = 0$

In this case, $p_0''(0) = -6p_0 + 6p_1 - 4p_0' - 2p_1' = 0$

$$\Rightarrow -6p_0 + 6p_1 = 4p_0' + 2p_1'$$

$$\Rightarrow 3(p_1 - p_0) = 2p_0' + p_1'$$

$$p_{n-1}''(1) = 6p_{n-1} - 6p_n + 2p_{n-1}' + 4p_n' = 0$$

$$\Rightarrow 3(p_n - p_{n-1}) = p_{n-1}' + 2p_n'$$