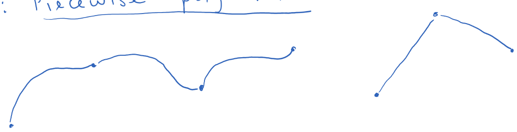
Splines 9/21/2021

Tuesday, September 21, 2021 12:40 PM

Last week, talked about interpolation (2 points). What about 2+ points?

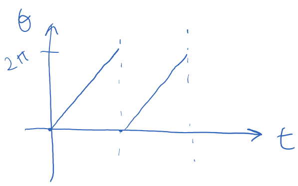
Answer: Piecewise polynomials



How should we handle the shape of each segment's polynomial, esp. how can we make the polynomial smooth at each point?

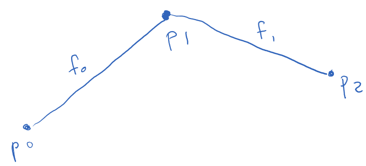
Aside: Continuity of curves

No Continuity



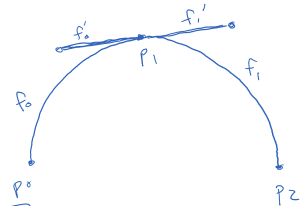
EX Increasing angles over time

C^0 continuity



EX Linear interpolation between data pts

C^1 continuity



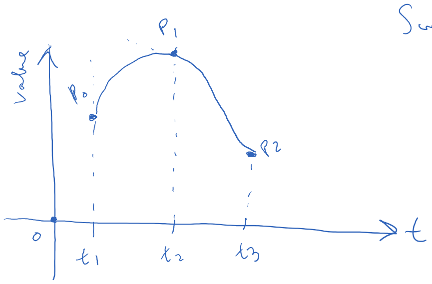
EX Catmull-Rom Spline

Spline: piecewise polynomial connect multiple data points

Catmull-Rom: Cubic curves interpolate each segment. Given data pts, other ctrl pts automatically generated

Spline Example:

EX



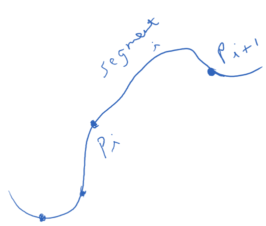
Suppose the user gives us 3 keys

keys = $[\langle t_1, p_0\rangle, \langle t_2, p_1\rangle, \langle t_3, p_2\rangle]$

ctrl points: $[\underbrace{b_0\ b_1\ b_2^1\ b_3}_{\text{segment } 0} \mid \underbrace{b_0^2\ b_1^2\ b_2^2\ b_3^2}_{\text{segment } 1}]$

(assume cubic)

NOTE: If we're using cubic Bezier curves for each segment i, then $b_0^i = p_i$ & $b_3^i = p_{i+1}$



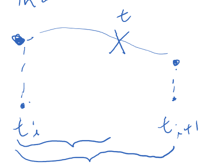
NOTE: If all times are uniform, e.g. $t_1 = 1$, $t_2 = 2$, $t_3 = 3$, etc., we do not store the time in this case.

NOTE: The curve between p_i & p_{i+1} is parameterized by $u \in [0,1]$

... $p(t)$.

→ To get a value $p(t)$:

1. find the interval $[t_i, t_{i+1}]$ that contains t



2. compute $u = \dfrac{t - t_i}{t_{i+1} - t_i}$

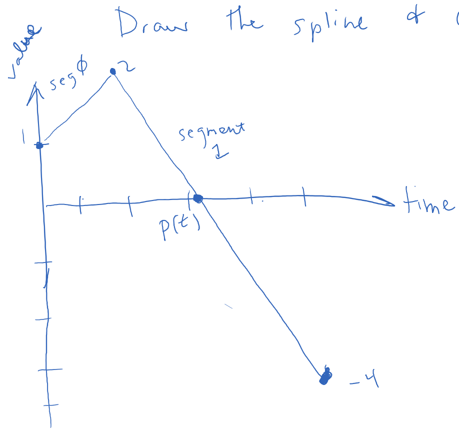
3. Get the ctrl pts for segment i:
   $b_0^i \quad b_1^i \quad b_2^i \quad b_3^i$

4. Return $p = \text{interpolate}(b_0, b_1, b_2, b_3, u)$

→ What if $t < t_1$ (before first key) or $t > t_n$ (after last key)?

→ clamp
→ loop } most common
→ "extrapolate" (keep going based on endpt slope)

Ex Suppose we have keys $\overset{t_1\ p_0}{\langle 0, 1 \rangle}, \overset{t_2\ p_1}{\langle 2, 2 \rangle}, \overset{t_3\ p_2}{\langle 5, -4 \rangle}$
& linearly interpolate between values.
Draw the spline & compute the value ~~at~~ $t = 3$.

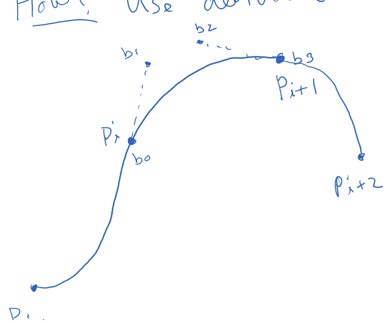


1. We are in segment #1 at $t = 3$
2. $u = \dfrac{t - t_i}{t_{i+1} - t_i} = \dfrac{3 - 2}{5 - 2} = \dfrac{1}{3}$
3. Control points are just p_1 & p_2
4. $p = p_1(1 - t) + p_2 t$
   $= 2\left(1 - \frac{1}{3}\right) + (-4)\left(\frac{1}{3}\right)$
   $= 2\left(\frac{2}{3}\right) - 4/3$
   $= 0$

Catmull-Rom Splines:

Idea: Given keys $p_0\, p_1 \ldots p_n$, compute the ctrl pts s.t. the slopes have C^1 continuity

How? Use derivative at each p_i to get positions for b_1 & b_2



$b_0 = p_i$

$b_1 = p_i + \frac{1}{3}\left(\dfrac{p_{i+1} - p_{i-1}}{2}\right) = p_i + \frac{1}{6}\left(p_{i+1} - p_{i-1}\right)$

$b_2 = p_{i+1} - \frac{1}{3}\left(\dfrac{p_{i+2} - p_i}{2}\right) = p_{i+1} - \frac{1}{6}\left(p_{i+2} - p_i\right)$

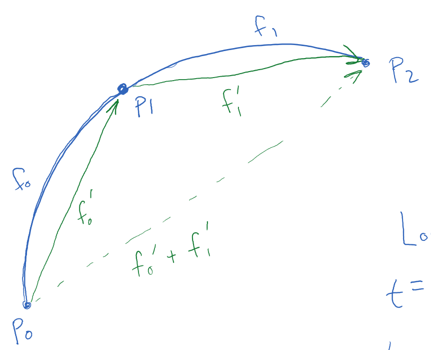
$b_3 = p_{i+1}$

For first segment, let $b_1 = b_0 + \frac{1}{6}(p_1 - p_0)$

For last segment, let $b_2 = b_3 - \frac{1}{6}(p_n - p_{n-1})$

Algebraic derivation / perspective :

Recall : $p(t) = (1-t)^3 b_0 + 3t(1-t)^2 b_1 + 3t^2(1-t) b_2 + t^3 b_3$



$p'(t) = \underline{-3(1-t)^2} b_0 + \underline{3(1-t)^2} b_1 - \underline{6t(1-t)} b_1 + \underline{6t(1-t)} b_2 - \underline{3t^2} b_2 + \underline{3t^2} b_3$

$= 3(1-t)^2(b_1 - b_0) + 6t(1-t)(b_2 - b_1) + 3t^2(b_3 - b_2)$

Look at what happens at $t=0$ & $t=1$

$t=0$ (begining of segment) $\rightarrow$ $p'(0) = 3(b_1 - b_0)$

$t=1$ (end of segment) $\longrightarrow$ $p'(1) = 3(b_3 - b_2)$

We see that first 2 ctrl pts control slope at begining of segment & last 2 ctrl pts control slope at end of segment.

But what should these slopes actually be?

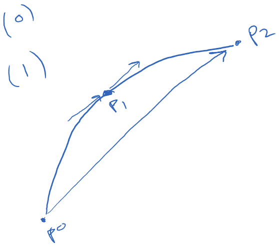
Approach: Use difference between p_1 & p_0 to estimate f_0'
(e.g. rate of change is the change in position divided by time)

$f_0' = \frac{p_1 - p_0}{\Delta t} = \frac{p_1 - p_0}{1}$

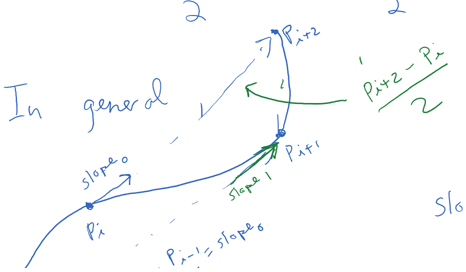
Similarly for $f_1' = \frac{p_2 - p_1}{1}$

We want $f_0' = f_1'$ $\Rightarrow$ take their average as the slope for both

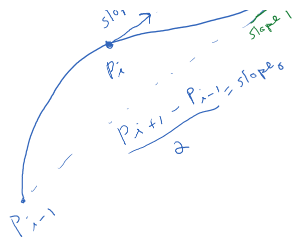
$\text{Slope} = \frac{f_0' + f_1'}{2} = \frac{(p_1 - p_0) + (p_2 - p_1)}{2} = \frac{p_2 - p_0}{2}$ $\leftarrow$ use this for both $p'(0)$ & $p'(1)$



In general



$\text{slope}_0 = \frac{p_{i+1} - p_{i-1}}{2}$



$$\text{slope}_0 = \frac{p_{i+1} - p_{i-1}}{2}$$

$$\text{slope}_1 = \frac{p_{i+2} - p_i}{2}$$

Putting it together, we have

$$p'(0) = \text{slope}_0 = \frac{p_{i+1} - p_{i-1}}{2} = 3(\underbrace{b_1}_{\text{unknown}} - \underbrace{b_0}_{p_i})$$

$$\Rightarrow \quad p_{i+1} - p_{i-1} = 6(b_1 - p_i)$$

$$p_{i+1} - p_{i-1} = 6b_1 - 6p_i$$

$$6p_i + p_{i+1} - p_{i-1} = 6b_1$$

$$p_i + \frac{1}{6}(p_{i+1} - p_{i-1}) = b_1$$

$$p'(1) = \text{slope}_1 = \frac{p_{i+2} - p_i}{2} = 3(\underbrace{b_3}_{p_{i+1}} - \underbrace{b_2}_{\text{unknown}})$$

$$\Rightarrow \quad p_{i+2} - p_i = 6p_{i+1} - 6b_2$$

$$6b_2 = 6p_{i+1} - p_{i+2} + p_i$$

$$b_2 = p_{i+1} - \frac{1}{6}(p_{i+2} - p_i)$$