Lecture 9/16/2021

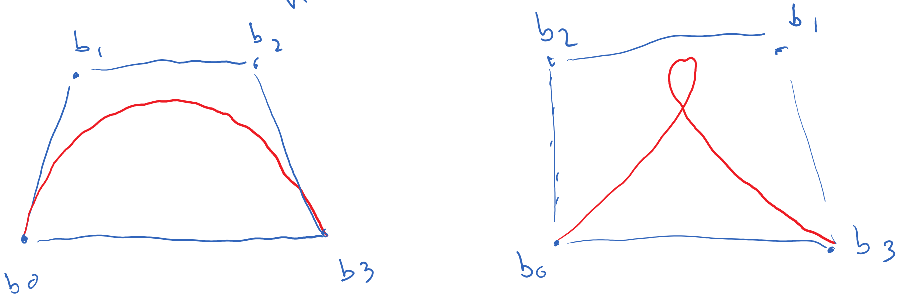
Thursday, September 16, 2021 12:07 PM

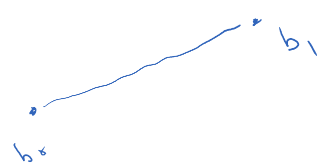
Similarities between linear & cubic interpolation

1. The coefficients ("t" terms) sum to one
2. The # ctrl pts = degree + 1
3. p(0) gives the first control pt
   p(1) gives the last control pt
4. (less obvious) The curve is always w/in the region bounded by the control pts, for $t \in [0,1]$

convex hull

$\boxed{\text{EX}}$





Both linear & cubic interpolation curves are examples of Bezier curves.

→ any degree curve can be used to interpolate 2 points

An n^{th}-degree Bezier curve has the form

$$p(t) = \sum_{i=0}^{n} \underbrace{B_i^n(t)}_{\text{ex: terms w/ t}} \underbrace{b_i}_{\text{ctrl pt}} \quad \text{where} \quad \underbrace{B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}}_{\text{Bernstein Polynomial}}$$

ex: terms w/ t ... ith ... them

ex: terms with t^i in them

polyno

note: $\binom{n}{i} = \frac{n!}{i!\,(n-i)!}$

$\boxed{\text{EX}}$ Derive an expression for a degree-1 Bezier curve.

$$p(t) = \sum_{i=0}^{1} B_i^1(t)\, b_i = B_0^1(t)\, b_0 + B_1^1(t)\, b_1$$

$$= \binom{1}{0} t^0 (1-t)^{1-0}\, b_0 + \binom{1}{1} t^1 (1-t)^{1-1}\, b_1$$

$$= (1-t)\, b_0 + t\, b_1$$

note: $\binom{1}{0} = \frac{1!}{0!\,(1-0)!} = 1$

$\binom{1}{1} = \frac{1!}{1!\,(1-1)!} = 1$

note: $0! = 1$

Properties of Bernstein polynomials

① $B_i^n(t) \in [0,1]$ when $t \in [0,1]$

② $\sum_{i=0}^{n} B_i^n(t) = 1$

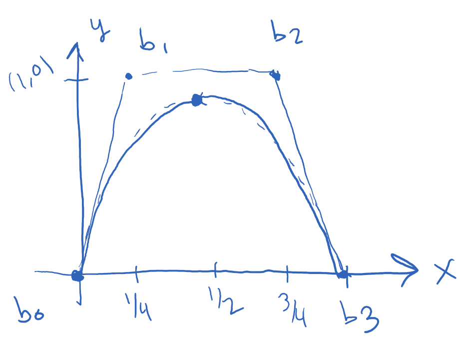
$\boxed{\text{Exercise}}$ Draw the ctrl pts & curve for the following points.

$b_0 = (0,0)^T$

$b_1 = (1/4, 1)^T$

$b_2 = (3/4, 1)^T$

$b_3 = (1,0)$



Compute value at $t = 1/2$

$$p(\tfrac{1}{2}) = (1-\tfrac{1}{2})^3 b_0 + 3(\tfrac{1}{2})(1-\tfrac{1}{2})^2 b_1 + 3(\tfrac{1}{2})^2(1-\tfrac{1}{2})\, b_2 + (\tfrac{1}{2})^3 b_3$$

$$= (\tfrac{1}{2})^3 b_0 + 3(\tfrac{1}{2})^3 b_1 + 3(\tfrac{1}{2})^3 b_2 + (\tfrac{1}{2})^3 b_3$$

$$= \frac{1}{8}\left[b_0 + 3b_1 + 3b_2 + b_3 \right]$$

$$= \frac{1}{8}\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1/4 \\ 1 \end{pmatrix} + 3\begin{pmatrix} 3/4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$= \frac{1}{8}\left[\begin{pmatrix}0\\0\end{pmatrix} + 3\begin{pmatrix}\ \\1\end{pmatrix} + \begin{pmatrix}\ \\1\end{pmatrix} \ldots\right]$

$= \frac{1}{8}\begin{bmatrix}3/4 + 9/4 + 4/4\\ 6\end{bmatrix} = \frac{1}{8}\begin{bmatrix}4\\6\end{bmatrix} = \begin{bmatrix}1/2\\3/4\end{bmatrix}$

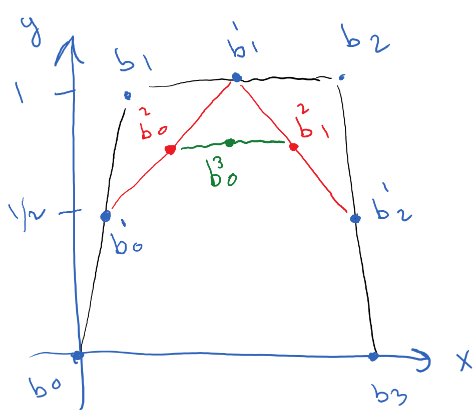
Implementation:

How to compute $p(t)$ given t.

1. Direct computation of $p(t)$
2. De casteljau's algorithm

De Casteljau's Algorithm:

Consider the same curve as before. Use de casteljau's alg. to compute $p(\frac{1}{2})$



Notation:

$\mathrm{LERP}(b_i, b_{i+1}, t)$ ← linear interpolation

$= b_i(1-t) + b_{i+1}\,t$

b_0^1: 1 ← "level", 0 ← "start of lerp"

$$\underbrace{\begin{matrix}\mathrm{LERP}(b_0, b_1, t) = b_0^1\\ \mathrm{LERP}(b_1, b_2, t) = b_1^1\\ \mathrm{LERP}(b_2, b_3, t) = b_2^1\end{matrix}}_{\text{"level 1"}} \;\Big\rangle\; \underbrace{\begin{matrix}\mathrm{LERP}(b_0^1, b_1^1, t) = b_0^2\\ \mathrm{LERP}(b_1^1, b_2^1, t) = b_1^2\end{matrix}}_{\text{"level 2"}} \;\Big\rangle\; \underbrace{\mathrm{LERP}(b_0^2, b_1^2, t) = b_0^3 \leftarrow p(t)}_{\text{"level 3"}}$$

Level 1:

$b_0(1-t) + b_1 t = \begin{pmatrix}0\\0\end{pmatrix}\frac{1}{2} + \begin{pmatrix}1/4\\1\end{pmatrix}\frac{1}{2} = \begin{pmatrix}1/8\\1/2\end{pmatrix}$

$\ldots\begin{pmatrix}3/4\\ \ \end{pmatrix}\frac{1}{2} = \begin{pmatrix}1/2\\ \ \end{pmatrix}$

$b_1^1 = b_1(1-t) + b_2 t = \begin{pmatrix} 1/4 \\ 1 \end{pmatrix} \frac{1}{2} + \begin{pmatrix} [illegible] \\ 1 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} [illegible] \\ 1 \end{pmatrix}$

$b_2^1 = b_2(1-t) + b_3 t = \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} \frac{1}{2} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} 7/8 \\ 1/2 \end{pmatrix}$

Level 2:

$b_0^2 = \begin{pmatrix} 1/8 \\ 1/2 \end{pmatrix} \frac{1}{2} + \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} 5/16 \\ 3/4 \end{pmatrix}$

$b_1^2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \frac{1}{2} + \begin{pmatrix} 7/8 \\ 1/2 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} 11/16 \\ 3/4 \end{pmatrix}$

Level 3:

$b_0^3 = \begin{pmatrix} 5/16 \\ 3/4 \end{pmatrix} \frac{1}{2} + \begin{pmatrix} 11/16 \\ 3/4 \end{pmatrix} \frac{1}{2} = \begin{pmatrix} 16/32 \\ 6/8 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 3/4 \end{pmatrix}$

Advantages & Disadvantages:

de Casteljau

+ simple
+ generalizes any degree Bezier curve
+ numerically stable

Direct computation w/ Bernstein polynomials

- powers of small values numerically unstable
+ generalizes to any degree Bezier
+ easy to take derivatives

Drawing curves

Implement piecewise linear curve

```
for ( float t=0; t<1; t ± step)
    a = p(t)
    b = p(t+step)
    drawLine(a,b)
```

More Properties of Bezier Curves:

Affine Invariance: to translate, rotate, scale a curve, we only need to modify the ctrl pts

Linear Precision: when all ctrl pts are co-linear, the curve is a line

Morphs & Animation:

to animate a curve, animate its ctrl pts